

SYMMETRIES OF TODA EQUATIONS

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ABSTRACT

We find a sequence consisting of time dependent evolution vector fields whose time independent part corresponds to the master symmetries for the Toda equations. Each master symmetry decomposes as a sum consisting of a group symmetry and a Hamiltonian vector field. Taking Lie derivatives in the direction of these vector fields produces an infinite sequence of recursion operators.

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1. Introduction: A symmetry group of a system of differential equations is a Lie group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. The symmetry approach to solving differential equations can be found, for example, in the books of Olver [16], Bluman and Cole [2], Bluman and Kumei [3], and Ovsiannikov [17]. One method of finding symmetry groups is the use of recursion operators, an idea introduced by Olver [15]. The existence of a recursion operator provides a mechanism for generating infinite hierarchies of symmetries. Most of the well known integrable equations, including the KdV, do have a recursion operator. Even some non conservative systems have recursion operators. The Toda Lattice is one example where a recursion operator is not known. In [6] we used master symmetries to generate nonlinear Poisson brackets for the Toda Lattice. In essence, it is an example of a system which is not only bihamiltonian but it can actually be given N different Hamiltonian formulations with N as large as we please. In most cases, if a system is bihamiltonian, one can find a recursion operator by inverting one of the Poisson operators. However in the case of Toda Lattice both operators are non-invertible and therefore this method fails. Master symmetries were first introduced by Fokas and Fuchssteiner in [8] in connection with the Benjamin-Ono Equation. Then in W. Oevel and B. Fuchssteiner [14] a master symmetry was found for the Kadomtsev-Petviashvili equation. Master symmetries for equations in $1 + 1$, like the KdV, are discussed in Chen, Lee and Lin [4] and in Fokas [9]. General theory of master symmetries is discussed in Fuchssteiner [11]. Connection between master symmetries and usual recursion operators for equations in $2 + 1$ is discussed in [10]. Some properties of master symmetries (at least in the Toda case) are clear: They preserve constants of motion, Hamiltonian vector fields and they generate a hierarchy of Poisson brackets. We are interested in the following problem : Can one find a symmetry group of the system whose infinitesimal generator is a given master symmetry? In other words, is a master symmetry a group symme-

try? In the case of Toda equations the answer is negative. However, in this paper we find a sequence consisting of time dependent evolution vector fields whose time independent part corresponds to the master symmetries in [6]. Each master symmetry X_n can be written in the form $Y_n + tZ_n$ where Y_n is a time dependent symmetry and Z_n is time independent Hamiltonian symmetry (i.e. a Hamiltonian vector field). Taking Lie derivatives in the direction of X_n (or Y_n) gives an infinite sequence of recursion operators for the Toda Lattice.

2. In this section we present some background on the Toda lattice. See [7] for more details. We also include some of the results in [6] for completeness.

The Toda lattice is a Hamiltonian system with Hamiltonian

$$H(q_1, \dots, q_N, p_1, \dots, p_N) = \sum_{i=1}^N \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}} .$$

This system is completely integrable. One can find a set of functions $\{H_1, \dots, H_N\}$ which are constants of motion for Hamilton's equations. To determine the constants of motion, we use Flaschka's transformation

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})} , \quad b_i = -\frac{1}{2} p_i .$$

Then

$$\dot{a}_i = a_i(b_{i+1} - b_i) \dot{b}_i = 2(a_i^2 - a_{i-1}^2) .$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix} ,$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & & -a_{N-1} & 0 \end{pmatrix}.$$

It follows easily that the eigenvalues of L do not evolve with time.

In [5] [6] we constructed a sequence of vector fields X_n , for $n \geq -1$, and an infinite sequence of contravariant 2-tensors w_n , for $n \geq 1$, satisfying :

- i) w_n are all Poisson.
- ii) The functions $H_n = \frac{1}{n} \text{Tr } L^n$ are in involution with respect to all of the w_n .
- iii) $X_n(H_m) = (n+m)H_{n+m}$.
- iv) $L_{X_n} w_m = (n-m+2)w_{n+m}$, modulo an equivalence relation defined in [].
- v) $[X_n, \chi_l] = (l-1)\chi_{l+n}$, where χ_l is the Hamiltonian vector field generated by H_l with respect to w_1 .
- vi) $M_n \text{ grad } H_l = M_{n-1} \text{ grad } H_{l+1}$, where M_n is the Poisson matrix of w_n . If we denote the Hamiltonian vector field of H_l with respect to the n th bracket by χ_l^n , then these relations are equivalent to $\chi_l^n = \chi_{l+1}^{n-1}$.

We give an outline of the construction of the vector fields X_n . We define X_{-1} to be

$$\text{grad } H_1 = \text{grad } \text{Tr } L = \sum_{i=1}^N \frac{\partial}{\partial b_i}$$

and X_0 to be the Euler vector field

$$\sum_{i=1}^{N-1} a_i \frac{\partial}{\partial a_i} + \sum_{i=1}^N b_i \frac{\partial}{\partial b_i}.$$

We want X_1 to satisfy

$$X_1(\text{Tr } L^n) = n \text{Tr } L^{n+1} .$$

One way to find such a vector field is by considering the equation

$$\dot{L} = [B, L] + L^2 . \quad (1)$$

Note that the left hand side of this equation is a tridiagonal matrix while the right hand side is pentadiagonal. We look for B as a tridiagonal matrix

$$B = \begin{pmatrix} \gamma_1 & \beta_1 & 0 & \cdots & \cdots \\ \alpha_1 & \gamma_2 & \beta_2 & \cdots & \cdots \\ 0 & \alpha_2 & \gamma_3 & \beta_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} . \quad (2)$$

We want to choose the α_i , β_i and γ_i so that the right hand side of equation (9) becomes tridiagonal. One simple solution is $\alpha_n = -(n+1)a_n$, $\beta_n = (n+1)a_n$, $\gamma_n = 0$. The vector field X_1 is defined by the right hand side of (9) and :

$$X_1 = \sum_{n=1}^{N-1} \dot{a}_n \frac{\partial}{\partial a_n} + \sum_{n=1}^N \dot{b}_n \frac{\partial}{\partial b_n} , \quad (3)$$

where

$$\begin{aligned} \dot{a}_n &= -na_nb_n + (n+2)a_nb_{n+1} \\ \dot{b}_n &= (2n+3)a_n^2 + (1-2n)a_{n-1}^2 + b_n^2 . \end{aligned} \quad (4)$$

To construct the vector field X_2 we consider the equation

$$\dot{L} = [B, L] + L^3 . \quad (5)$$

The calculations are similar to those for X_1 . The matrix B is now pentadiagonal and the system of equations slightly more complicated. The result is a vector field

$$X_2 = \sum_{n=1}^{N-1} \dot{a}_n \frac{\partial}{\partial a_n} + \sum_{n=1}^N \dot{b}_n \frac{\partial}{\partial b_n} \quad (6)$$

where

$$\begin{aligned}
\dot{a}_n &= (2-n)a_{n-1}^2 a_n + (1-n)a_n b_n^2 + a_n b_n b_{n+1} + \\
&\quad (n+1)a_n a_{n+1}^2 + (n+1)a_n b_{n+1}^2 + a_n^3 + \sigma_n a_n (b_{n+1} - b_n) \\
\dot{b}_n &= 2\sigma_n a_n^2 - 2\sigma_{n-1} a_{n-1}^2 + (2n+2)a_n^2 b_n + (2n+1)a_n^2 b_{n+1} + \\
&\quad + (3-2n)a_{n-1}^2 b_{n-1} + (4-2n)a_{n-1}^2 b_n + b_n^3,
\end{aligned} \tag{7}$$

with

$$\sigma_n = \sum_{i=1}^{n-1} b_i \tag{8}$$

and $\sigma_1 = 0$.

For $n \geq 3$ we define X_n by

$$[X_1, X_{n-1}] = (n-2)X_n. \tag{9}$$

We consider X_n as an equivalence class of vector fields. We define $X_n \sim Y_n$ if $X_n - Y_n = k\chi_{n+1}$, for some real number k . It can be easily shown that $[X_i, X_j]$ is equivalent to $(j-i)X_{i+j}$ for $i, j \geq 0$. Moreover, we believe, but we don't have a proof, that the two expressions are actually equal (not just equivalent).

3. In this section we find an infinite sequence of evolution vector fields that are symmetries of equations (3). We do not know if every symmetry of Toda equations is included in this sequence.

We begin by writing equations (3) in the form

$$\begin{aligned}
\Gamma_j &= \dot{a}_j - a_j b_{j+1} + a_j b_j = 0 \\
\Delta_j &= \dot{b}_j - 2a_j^2 + 2a_{j-1}^2 = 0.
\end{aligned} \tag{10}$$

We look for symmetries of Toda equations. i.e. vector fields of the form

$$\mathbf{v} = \tau \frac{\partial}{\partial t} + \sum_{j=1}^{N-1} \phi_j \frac{\partial}{\partial a_j} + \sum_{j=1}^N \psi_j \frac{\partial}{\partial b_j} \tag{11}$$

that generate the symmetry group of the Toda System. The first

prolongation of \mathbf{v} is

$$\text{pr}^{(1)}\mathbf{v} = \mathbf{v} + \sum_{j=1}^{N-1} f_j \frac{\partial}{\partial \dot{a}_j} + \sum_{j=1}^N g_j \frac{\partial}{\partial \dot{b}_j} , \quad (12)$$

where

$$\begin{aligned} f_j &= \dot{\phi}_j - \dot{\tau} \dot{a}_j \\ g_j &= \dot{\psi}_j - \dot{\tau} \dot{b}_j . \end{aligned} \quad (13)$$

The infinitesimal condition for a group to be a symmetry of the system is

$$\begin{aligned} \text{pr}^{(1)}(\Gamma_j) &= 0 \\ \text{pr}^{(1)}(\Delta_j) &= 0 . \end{aligned} \quad (14)$$

Therefore we obtain the equations

$$\begin{aligned} \dot{\phi}_j - \dot{\tau} a_j (b_{j+1} - b_j) + \phi_j (b_j - b_{j+1}) + a_j \dot{\psi}_j - a_j \dot{\psi}_{j+1} &= 0 \\ \dot{\psi}_j - 2\dot{\tau} (a_j^2 - a_{j-1}^2) - 4a_j \dot{\phi}_j + 4a_{j-1} \dot{\phi}_{j-1} &= 0 . \end{aligned} \quad (15)$$

We first give some obvious solutions :

i) $\tau = 0$, $\phi_j = 0$, $\psi_j = 1$. This is the vector field X_{-1} .

ii) $\tau = -1$, $\phi_j = 0$, $\psi_j = 0$. The resulting vector field is the time translation $-\frac{\partial}{\partial t}$ whose evolutionary representative is

$$\sum_{j=1}^{N-1} \dot{a}_j \frac{\partial}{\partial a_j} + \sum_{j=1}^N \dot{b}_j \frac{\partial}{\partial b_j} . \quad (16)$$

This is the Hamiltonian vector field χ_{H_2} . It generates a Hamiltonian symmetry group.

iii) $\tau = -1$, $\phi_j = a_j$, $\psi_j = b_j$. Then

$$\mathbf{v} = -\frac{\partial}{\partial t} + \sum_{j=1}^{N-1} a_j \frac{\partial}{\partial a_j} + \sum_{j=1}^N b_j \frac{\partial}{\partial b_j} = -\frac{\partial}{\partial t} + X_0 . \quad (17)$$

This vector field generates the same symmetry as the evolutionary vector field

$$X_0 + t\chi_{H_2} . \quad (18)$$

We next look for some non obvious solutions. The vector field X_1 is not a symmetry, so we add a term which depends on time. We try

$$\begin{aligned}\phi_j &= -ja_j b_j + (j+2)a_j b_{j+1} + \\ &\quad t(a_j a_{j+1}^2 + a_j b_{j+1}^2 - a_{j-1}^2 a_j - a_j b_j^2) \\ \psi_j &= (2j+3)a_j^2 + (1-2j)a_{j-1}^2 + b_j^2 + \\ &\quad t(2a_j^2 b_{j+1} + 2a_j^2 - 2a_{j-1}^2 a_j - 2a_{j-1}^2 b_j)\end{aligned}\tag{19}$$

and $\tau = 0$.

A tedious but straightforward calculation shows that ϕ_j, ψ_j satisfy (23). It is also straightforward to check that the vector field $\sum \phi_j \frac{\partial}{\partial a_j} + \sum \psi_j \frac{\partial}{\partial b_j}$ is precisely equal to $X_1 + t\chi_{H_3}$. The pattern suggests that $X_n + t\chi_{H_{n+2}}$ is a symmetry of Toda equations. In the course of the proof we use some properties of the first three Poisson brackets w_1, w_2, w_3 of the Toda lattice. These three brackets have been known for some time [1], [13].

Theorem *The vector fields $X_n + t\chi_{H_{n+2}}$ are symmetries of Toda equations for $n \geq -1$.*

Proof : Note that $\chi_{H_1} = 0$ because H_1 is a Casimir for the Lie-Poisson w_1 bracket. We first prove the formula

$$[X_1, \chi_l] = (l-1)\chi_{l+1} . \tag{20}$$

We write $\chi_l = [w_1, H_l]$ where $[,]$ denotes the Schouten bracket. We use the super Jacobi identity for the Schouten bracket.

$$[[w_1, H_l], X_1] + [[H_l, X_1], w_1] + [[X_1, w_1], H_l] = 0 . \tag{21}$$

Therefore,

$$\begin{aligned}[X_1, \chi_l] &= (l+1)[H_{l+1}, w_1] - 2[w_2, H_l] \\ &= (l+1)\chi_{l+1} - 2\chi_l^2 .\end{aligned}\tag{22}$$

But $\chi_l^2 = \chi_{l+1}^1 = \chi_{l+1}$. This is a Lenard type relation which is easily checked. See [5] for details. Therefore, we have

$$[X_1, \chi_l] = (l-1)\chi_{l+1} . \tag{23}$$

In the same fashion one can prove that

$$[X_2, \chi_l] = (l-1)\chi_{l+2} . \quad (24)$$

To prove it, one uses the relation $\chi_l^3 = \chi_{l+1}^2 = \chi_{l+2}$. For $n \geq 3$ we use induction on n .

$$\begin{aligned} [X_{n+1}, \chi_l] &= \left[\frac{1}{n-1} [X_1, X_n], \chi_l \right] \\ &= -\frac{1}{n-1} \{ [[X_n, \chi_l], X_1] - [[\chi_l, X_1], X_n] \} \\ &= \frac{1}{n-1} \{ [X_1, \chi_{n+l}] + (l-1)[\chi_{l+1}, X_n] \} \\ &= \frac{l-1}{n-1} [(l+n-1)\chi_{l+n+1} - l\chi_{l+n+1}] \\ &= (l-1)\chi_{l+n+1} . \end{aligned} \quad (25)$$

In particular, for $l = 2$, we have $[X_n, \chi_2] = \chi_{n+2}$.

Since the Toda flow is Hamiltonian, generated by χ_2 , to show that $Y_n = X_n + t\chi_{n+2}$ are symmetries of Toda equations we must verify the equation

$$\frac{\partial Y_n}{\partial t} + [\chi_2, Y_n] = 0 . \quad (26)$$

But

$$\begin{aligned} \frac{\partial Y_n}{\partial t} + [\chi_2, Y_n] &= \frac{\partial Y_n}{\partial t} + [\chi_2, X_n + t\chi_{n+2}] \\ &= \chi_{n+2} - [\chi_n, \chi_2] \\ &= \chi_{n+2} - \chi_{n+2} = 0 . \end{aligned} \quad (27)$$

□

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